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Fourier-Like Analysis

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ABSTRACT

Fourier transform, its applications and also its generalization are of interests of harmonic analysts. In this paper, we give a new generalization of this transform on n-dimensional Euclidean space $\mathbb{R}^n$. For this, we define an integral transform on $L^1(\mathbb{R}^n)$ by replacing the ordinary inner product in n-dimensional Fourier transform with a quadratic form and prove a reconstruction formula. The set of all linear operators on $\mathbb{R}^n$ preserving this quadratic form with determinant 1 are the indefinite special orthogonal group. We apply this group instead of ordinary rotation group to perform our new transform. This transform extends to an isometry on $L^2(\mathbb{R}^n)$. The convergence of n-dimensional trigonometric Fourier-like series are studied.

Keywords: Fourier-like transform, Integral transform, Quadratic form.

1. Introduction

We begin by introducing some notations that will appeare in this paper. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of positive integers, integers, real numbers and complex numbers, respectively. For any positive integer n, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space. An element of $\mathbb{R}^n$ is written as $x = (x_1, x_2, \dots, x_n)$, where each x_i, $i = 1, 2, \dots, n$ is a real number.

The Euclidean inner product of $x, y \in \mathbb{R}^n$ is the number $x.y = \sum_{i=1}^{n} x_i y_i$ and the norm of $x \in \mathbb{R}^n$ is the nonnegative number $|x| = \sqrt{x.x}$. Furthermore, $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$ denotes the ordinary Lebesgue measure on $\mathbb{R}^n$.
We will deal with some function spaces defined on $\mathbb{R}^n$. The simplest of them are the $L^p = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the spaces of all measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$ whose absolute value raised to the p-th power has finite integral, or equivalently, that $\| f \|_p = (\int_{\mathbb{R}^n} |f(x)|^p \; \mathrm{d}x)^{\frac{1}{p}} < \infty$.

The number $\| f \|_p$ is called the L^p norm of f. The space $L^\infty(\mathbb{R}^n)$ consists of all essentially bounded measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$ and for $f \in L^\infty(\mathbb{R}^n)$, $\| f \|_\infty$ is the essential supremum of $|f|$.

The general linear group $GL_n(\mathbb{R})$ consists of all invertible n-by-n matrices over $\mathbb{R}^n$. The orthogonal group of degree n is a subgroup of $GL_n(\mathbb{R})$ given by

$$O(n) = \{Q \in GL_n(\mathbb{R}) : Q^t Q = QQ^t = I_n\},$$

where Q^t is the transpose of Q and I_n denotes the identity matrix on $\mathbb{R}^n$.

Every orthogonal matrix has determinant either 1 or -1. The orthogonal n-by-n matrices with determinant 1 form a normal subgroup of $O(n)$ known as the special orthogonal group, $SO(n)$. The indefinite orthogonal group ,$O(p, q)$, with $p + q = n$ and $p, q \in \mathbb{N}$, is defined by

$$O(p,q) = \{A \in GL_n(\mathbb{R}) : A^t I_{p,q} A = I_{p,q}\},$$

with $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Note that for $A \in O(p,q)$, $\det A = \pm 1$.

The special indefinite orthogonal group is defined by $SO(p,q)=\{A\in O(p,q):\det A=1\}$. An integral transform is a mathematical operator that produces a new function $F(y)$ by integrating the product of an existing function $f(x)$ and a so-called kernel function $K(x, y)$ on a suitable interval I. The process, which is called transformation, is symbolized by the equation $F(y)=\int_I K(x,y)\,f(x)\,dx.$ Several such transforms exist and commonly named subject to the mathematicians who introduced them, such as the Laplace transform with the kernel e^{-xy} and $I=(0,+\infty)$; the Fourier transform with the kernel $\frac{1}{\sqrt{2\pi}}e^{-ixy}$ and $I=\mathbb{R}.$

Integral transforms are valuable for the simplification that they provide, most often in dealing with differential equations subject to particular boundary conditions. Proper choice of the class of transformation usually makes it possible to convert not only the derivatives in an intractable differential equation but also the boundary values into the terms of an algebraic equation that can be easily solved. The solution obtained is, of course, the transform of the solution of the original differential equation, and it is necessary to invert this transform to complete the operation; see Folland (1995), Debnath and Bhatta (2007) and Thomas (1999) for more details and examples.

If $f\in L^1(\mathbb{R}^n)$, the Fourier transform (FT) of f is the function $\hat{f}$ defined by

$$\hat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)\,e^{-ix.\xi}\,\mathrm{d}x,\quad \xi\in\mathbb{R}^n.$$

Fourier Transform Basics: Let $f,g\in L^1(\mathbb{R}^n),\ \ b\in\mathbb{R}^n, a>0$ and $\rho\in SO(n)$. We have

function	FT
$f(x)$	$\hat{f}(\xi)$
$f(x-b)$	$e^{-ib.\xi}\hat{f}(\xi)$
$e^{ix.b}f(x)$	$\hat{f}(y-b)$
$\frac{1}{\sqrt{a^n}}f(\frac{x}{a})$	$\sqrt{a^n}\hat{f}(a\xi)$
$f(\rho^{-1}y)$	$\hat{f}(\rho^{-1}y)$
$f*g(x)$	$(2\pi)^{n/2}\hat{f}(\xi)\hat{g}(\xi)$
$\overline{f(-x)}$	$\overline{\hat{f}(y)}$

where $f*g$ is the convolution of f and g defined by

$$(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y\ ,\quad x\in\mathbb{R}^n.$$

The inverse Fourier transform on L^1 is defined by setting $\breve{f}(x)=\hat{f}(-x)$ for $f\in L^1(\mathbb{R}^n)$ and $(\hat{f})\breve{}=f$ almost everywhere when $\hat{f}$ is also integrable. The Fourier transform is an L^2 isometry on $L^1\cap L^2$, which is a dense subspace of L^2. By density, there is a unique bounded extension of Fourier transform on L^2 that is also an isometry on L^2 (This is sometimes called the Plancherel Theorem). For more details see Rudin (1986) and Grafakos (2008).

In the present paper, we define a new integral transform on $L^1(\mathbb{R}^n)$ by replacing the Euclidean inner product in the kernel function of Fourier transform with a quadratic form and study its properties. This transform extends to an isometry on $L^2(\mathbb{R}^n)$. Finally, we approximate a function with its trigonometric Fourier-Like series and prove a convergence theorem.

2. The Fourier-Like Integral Formula

Let $n,p,q\in\mathbb{N}$, with $p+q=n$ and consider the following quadratic form,

$$[x,y]=x_1y_1+x_2y_2+\cdots+x_py_p-x_{p+1}y_{p+1}-\cdots-x_{p+q}y_{p+q},\qquad x,y\in\mathbb{R}^n.$$

Definition 2.1. Given f in $L^1(\mathbb{R}^n)$, we define

$$\tilde{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i[x,y]} \, \mathrm{d}x, \quad y \in \mathbb{R}^n.$$

We call $\tilde{f}$ the *Fourier-like transform* (F-L T) of f.

When a function $f(x_1, x_2, \cdots, x_n)$ of *n* variables can be written as a product of *n* functions of one variable, as in

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n),$$

its Fourier-like transform can be calculated as

$$\tilde{f}(y_1, y_2, \ldots, y_n) = f_1(y_1) f_2(y_2) \cdots f_p(y_p) \breve{f}_{p+1}(y_{p+1}) \cdots \breve{f}_n(y_n).$$

If $f \in L^1(\mathbb{R}^p)$ and $g \in L^1(\mathbb{R}^q)$, the function $f \otimes g$ on $\mathbb{R}^n$ defined by

$$f \otimes g(x_1, x_2) = f(x_1) g(x_2), \quad x_1 \in \mathbb{R}^p, \ x_2 \in \mathbb{R}^q,$$

is called the tensor product of f and g.

According to what was stated above, we obtain

$$(f \otimes g)^{\tilde{}} = \hat{f} \, \breve{g}.$$

Example 2.2. If $f = e^{-|x|^2}$, then $\tilde{f}(y) = \frac{1}{\sqrt{2^n}} \; e^{-\frac{|y|^2}{4}}$.

Example 2.3. If $f(x_1, x_2, \ldots, x_n) = e^{-|x_1| - |x_2| - \cdots - |x_n|}$, then

$$\tilde{f}(y_1, y_2, \ldots, y_n) = (\frac{2}{\pi})^{\frac{n}{2}} (\frac{1}{1+y_1^2})(\frac{1}{1+y_2^2}) \cdots (\frac{1}{1+y_n^2}).$$

Theorem 2.4.

a. The mapping $f \mapsto \tilde{f}$ *is a bounded linear transformation from* $L^1(\mathbb{R}^n)$, *to* $L^\infty(\mathbb{R}^n)$. *In fact* $\|\tilde{f}\|_\infty \le \|f\|_1$.

b. (*Riemann-Lebesgue-like Lemma*) *If* $f \in L^1(\mathbb{R}^n)$, *then* $\tilde{f}$ *is continuous and* $\tilde{f}(y) \to 0$ *as* $|y| \to \infty$. *Thus* $\tilde{f} \in C_0$.

Proof. *a.* is obvious. If $y_m \to y,$ then

$$|\tilde{f}(y_m) - \tilde{f}(y)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)|\, |e^{-i[x,y_m]} - e^{-i[x,y]}|\, dx.$$

The integrand is bounded by $2|f(x)|$ and tends to 0 as $m \to \infty$. Hence $\tilde{f}$ is continuous by the Lebesgue dominated convergence theorem. The last part of theorem is obvious when f is the characteristic function of n-cell $I = \{x \in \mathbb{R}^n : a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \cdots, a_n \le x_n \le b_n\}.$ The same is therefore true for simple functions on $\mathbb{R}^n$ and so is for each $f \in L^1(\mathbb{R}^n)$. □

Proposition 2.5. (*Fourier-Like Transform Basics*). *Given f, g in* $L^1(\mathbb{R}^n)$, $a, a_1, a_2, \ldots, a_n > 0$, $b \in \mathbb{R}^n$ and $\rho \in SO(p,q)$. Then we have

function	F-L T
$f(x)$	$\tilde{f}(y)$
$f(x-b)$	$e^{-i[y,b]}\tilde{f}(y)$
$\frac{1}{\sqrt{a^n}} f(\frac{x}{a})$	$\sqrt{a^n}\,\tilde{f}(ay)$
$f(a_1x_1, a_2x_2, \ldots, a_nx_n)$	$\frac{1}{a_1a_2\cdots a_n}\tilde{f}(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_n}{a_n})$
$e^{i[x,b]}f(x)$	$\tilde{f}(y-b)$
$f(\rho^{-1}x)$	$\tilde{f}(\rho^{-1}y)$
$f * g(x)$	$(2\pi)^{n/2}\tilde{f}(y)\tilde{g}(y)$
$\overline{f(-x)}$	$\overline{\tilde{f}(y)}$

Proof. All the verifications are rutin. We just prove that the fourier-like transform converts convolutions to pointwise product. By the Fubini's theorem, we can change the order of integration so that

$$
\begin{aligned}
(f*g)\,\tilde{}\,(y) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f*g(x)e^{-i[x,y]}\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i[x,y]}\left[\int_{\mathbb{R}^n} f(x-z)g(z)\,\mathrm{dz}\right]\mathrm{d}y \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} g(z)e^{-i[z,y]}\left[\int_{\mathbb{R}^n} f(x-z)e^{-i[x-z,y]}\mathrm{d}x\right]\mathrm{dz} \\
&= (2\pi)^{n/2}\,\tilde{f}(y)\tilde{g}(y). \;\square
\end{aligned}
$$

3. Inversion Formula

Let $H(x)=e^{-|x_1|-|x_2|-\cdots-|x_n|}$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ and for $\lambda>0$, define

$$h_\lambda(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} H(\lambda t)e^{i[t,x]}\,\mathrm{d}t.$$

Then

$$h_\lambda(x)=\frac{1}{(2\pi)^n}\left(\frac{2\lambda}{\lambda^2+x_1^2}\right)\left(\frac{2\lambda}{\lambda^2+x_2^2}\right)\cdots\left(\frac{2\lambda}{\lambda^2+x_n^2}\right).$$

Remark 3.1. Since $\int_{\mathbb{R}^n} h_\lambda(x)\,\mathrm{d}\,x=1$ and $h_\lambda(x)=\frac{1}{\lambda^n}h_1(\frac{x}{\lambda})$, by Bachman *et al.* (2000) and Stein and Weiss (1971), the family $\{h_\lambda\}_{\lambda>0}$ is an approximate identity for $L^1(\mathbb{R}^n)$ and so

(1) If $1\le p<\infty$ and $f\in L^p$, then $\| f*h_\lambda - f\|_p\to 0$ as $\lambda\to 0$,

(2) If $g\in L^\infty$ and g is continuous at a point x, then $\lim_{\lambda\to 0}(g*h_\lambda)(x)=g(x)$.

Proposition 3.2. *If* $f\in L^1(\mathbb{R}^n)$, *then*

$$f*h_\lambda(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} H(\lambda t)\,\tilde{f}(t)\,e^{i[t,x]}\,\mathrm{d}t.$$

Proof. Applying Fubini's theorem, we have

$$
\begin{aligned}
f * h_\lambda(x) &= \int_{\mathbb{R}^n} f(x-y) h_\lambda(y)\,\mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x-y) \left[\int_{\mathbb{R}^n} H(\lambda t)\, e^{i[t,y]} \mathrm{d}t \right] \mathrm{d}y \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} H(\lambda t) \left[\int_{\mathbb{R}^n} f(x-y)\, e^{i[t,y]} \mathrm{d}y \right] \mathrm{d}t \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} H(\lambda t) \left[\int_{\mathbb{R}^n} f(y)\, e^{i[t,x-y]} \mathrm{d}y \right] \mathrm{d}t \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} H(\lambda t)\, \tilde{f}(t)\, e^{i[t,x]} \mathrm{d}t. \ \square
\end{aligned}
$$

Theorem 3.3. (*Inversion Formula*). *If both .f and $\tilde{f}$ are in $L^1(\mathbb{R}^n)$, then*

$$
f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \tilde{f}(x) e^{i[x,y]}\, \mathrm{d}y, \quad x \in \mathbb{R}^n,
$$

for almost every x.

Proof. Let

$$
g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \tilde{f}(x) e^{i[x,y]}\, \mathrm{d}y, \quad x \in \mathbb{R}^n.
$$

By Proposition 3.2.

$$
f * h_\lambda(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} H(\lambda t)\, \tilde{f}(t)\, e^{i[t,x]}\, \mathrm{d}t. \tag{1}
$$

The integrands on the right side of (1) are bounded by $|\tilde{f}(t)|$, and since $H(\lambda t) \to 1$ as $\lambda \to 0$, the right side of (1) converges to $g(x)$ by Lebesgue dominated convergence theorem. By Remark 3.1. the family $\{f * h_\lambda\}_{\lambda>0}$ converges to f in $L^p(\mathbb{R}^n)$ with L^p norm. Thus $\{f * h_\lambda\}_{\lambda>0}$ has a subfamily which converges pointwise almost everywhere to f. Hence $f(x) = g(x)$ for almost every x. $\square$

4. Plancherel-Like Theorem For Fourier-Like Transform

Here we extend the Fourier-like transform to an isometry on $L^2(\mathbb{R}^n)$.

Theorem 4.1. *If* $f \in L^1 \cap L^2$, *then* $\tilde{f} \in L^2$ *and* $\|\tilde{f}\|_2 = \|f\|_2$.

Proof. Let $f_0(x) = \overline{f(-x)}$. Then $g = f * f_0 \in L^1(\mathbb{R}^n)$ and by Proposition 2.5,

$$\tilde{g} = (2\pi)^{n/2}\,\tilde{f}\,\tilde{f_0} = (2\pi)^{n/2}\,\tilde{f}\,\overline{\tilde{f}} = (2\pi)^{n/2}|\tilde{f}|^2 .$$

So we have

$$\begin{aligned} g(x) = f * f_0(x) &= \int_{\mathbb{R}^n} f(x-y)\overline{f(-y)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(x+y)\overline{f(y)}\,\mathrm{d}y \\ &= \langle f(.+x), f\rangle, \end{aligned}$$

where $\langle\, ,\rangle$ is the inner product in the Hilbert space $L^2(\mathbb{R}^n)$. Therefore g is continuous and

$$|g(x)| \le \|f(.+x)\|_2 \|f\|_2 = \|f\|_2^2 .$$

Applying Proposition 3.2 we get

$$(g * h_\lambda)(0) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} H(\lambda t)\tilde{g}(t)\,\mathrm{d}t. \tag{2}$$

Since g is continuous and bounded, Remark 3.1 shows that

$$\lim_{\lambda\to 0}(g * h_\lambda)(0) = g(0) = \langle f, f\rangle = \|f\|_2^2. \tag{3}$$

Since $H(\lambda t)$ increases to 1 as $\lambda \to 0$, the monotone convergence theorem gives

$$\lim_{\lambda\to 0}\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} H(\lambda t)\tilde{g}(t)dt = \int_{\mathbb{R}^n} |\tilde{f}(t)|^2\,\mathrm{d}t. \tag{4}$$

Now (2), (3) and (4) show that $\tilde{f} \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = \|\tilde{f}\|_2$. □

Theorem 4.2. *The Fourier-like transform is an isometry operator on* $L^2(\mathbb{R}^n)$.

Proof. Put $Y=\{\tilde{f} : f \in L^1 \cap L^2\}$. It can be easily shown that Y is dense in L^2. Since both $L^1 \cap L^2$ and Y are dense in L^2, we can extend the Fourier-like transform to an isometry of L^2 onto L^2. □

Corollary 4.2. *If* $f, g \in L^2$, *then* $\langle f, g\rangle = \langle \tilde{f}, \tilde{g}\rangle$.

5. Fourier-Like Series

Let $\mathbb{Z}^n$ be the set of all *n*-tuples of integers. The function $f(x)$ on $\mathbb{R}^n$ is periodic of period 1 in each variable if

$$f(x+m) = f(x) \ \text{ for all } \ m \in \mathbb{Z}^n.$$

Theorem 5.1. *The set of functions* $\{e^{2\pi i[m,x]} : m \in \mathbb{Z}^n\}$, *is orthonormal in* $L^2([0,1]^n)$, *where*

$$[0,1]^n = \underbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}_{n-\text{times}}.$$

Proof. It is obvious. □

Theorem 5.2. *If* $f(x) = \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i[m,x]}$ *on the* $[0, 1]^n$, *then*

$$c_m = \int_{[0,1]^n} f(x) e^{-2\pi i[m,x]}\, dx. \tag{5}$$

Proof. It concludes from Theorem 5.1. □

Definition 5.3. Suppose f is 1- periodic and integrable over $[0, 1]^n$. The numbers c_m defined by (5) are called the *Fourier-like coefficients* of f, and the corresponding series

$$f(x) \sim \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i[m,x]}$$

is called the *Fourier-like series* of f.

Theorem 5.4. *If* f *is* 1- *periodic and integrable over* $[0, 1]^n$ *and the Fourier-like coefficients* c_m *are defined by* (5), *then*

$$\sum_{m\in\mathbb{Z}^n} |c_m|^2 \le \int_{[0,1]^n} |f(x)|^2 \, \mathrm{d}x.$$

Proof.

$$f(x) - \sum_{-N}^{N} c_m e^{2\pi i[m,x]}|^2 = (f(x) - \sum_{-N}^{N} c_m e^{2\pi i[m,x]})(\overline{f(x)} - \sum_{-N}^{N} \overline{c_m} e^{-2\pi i[m,x]}) \ |$$

$$= | f(x)|^2 - \sum_{-N}^{N} [c_m \overline{f(x)} e^{2\pi i[m,x]} + \overline{c_m} f(x) e^{-2\pi i[m,x]}]$$

$$+ \sum_{m,l=-N}^{N} c_m \overline{c_l} e^{-2\pi i[m-l,x]}.$$

Integrating on $[0, 1]^n$ implies that

$$0 \le \int_{[0,1]^n} | f(x) - \sum_{-N}^{N} c_m e^{2\pi i[m,x]}|^2 \, \mathrm{d}x$$

$$= \int_{[0,1]^n} | f(x)|^2 \, \mathrm{d}x - \sum_{-N}^{N} [c_m \overline{c_m} + \overline{c_m} c_m] + \sum_{-N}^{N} c_m \overline{c_m}$$

$$= \int_{[0,1]^n} | f(x)|^2 \, \mathrm{d}x - \sum_{-N}^{N} | c_m |^2$$

Letting $N \to \infty$, we obtain the desired result. □

Corollary 5.5. *If* $f(x) \sim \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i[m,x]}$, *then* $c_m \to o$ *as* $|m| \to \infty$.

Since the trigonometric polynomials form an algebra which separates points of [0, 1] and contains the constant function **1**, the trigonometric polynomials are dense in $C([0, 1]^n)$ by Stone-Weierstrass theorem. The density of $C([0, 1]^n)$ in $L^p([0,1]^n)$, $1 \le p < \infty$, implies that the trigonometry polynomials are dense in $L^p([0,1]^n)$. Hence every function with zero Fourier-like coefficients is 0 almost everywhere.

Theorem 5.6. *Let* $f(x) \sim \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i[m,x]}$. *Then*

(*i*) *If* $f \in L^p([0,1]^n), 1 < p < \infty$, *then the Fourier-like series of* f *converges to* f *almost everywhere.*

(*ii*) *If* $f \in C([0,1]^n)$, *then the Fourier-like series of* f *converges to* f *for all* x.

Proof. Since the Fourier-like coefficients of $f(x) - \sum_{m \in \mathbb{Z}^n} c_m e^{2\pi i [m,x]}$ are all 0, the desired results holds. □

6. Conclusion

We have shown that the properties of Fourier transform and Fourier series in n dimensional Euclidean space are satisfied if the Euclidean inner product in the kernel function of Fourier transform replace with a quadratic form $[x, y] = x_1 y_1 + x_2 y_2 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$, $x, y \in \mathbb{R}^n$.

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